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Signatures of prelocalized states in classically chaotic systems

A. Ossipov, Tsampikos Kottos, and T. Geisel
Max-Planck-Institut für Strömungsforschung und Fakultät Physik der Universität Göttingen, Bunsenstraße 10, D-37073 Göttingen, Germany

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We investigate the statistics of eigenfunction intensities $\mathcal{P}(|\psi|^2)$ in dynamical systems with classical chaotic diffusion. Our results contradict some recent theoretical considerations that challenge the applicability of field theoretical predictions, derived in a different framework for diffusive disordered samples. For two-dimensional systems, the tails of $\mathcal{P}(|\psi|^2)$ contradict the results of the optimal fluctuation method, but agree very well with the predictions of the nonlinear σ model.

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The statistical properties of wave function intensities have sparked a great deal of research activity in recent years. These studies are not only relevant for mesoscopic physics [1–10], but also for understanding phenomena in areas of physics, ranging from nuclear [11] and atomic [12] to microwave physics [13] and optics [14]. Experimentally, using microwave cavity techniques it is possible to probe the microscopic structure of electromagnetic wave amplitudes in chaotic or disordered cavities [13]. Recently, the interest in this problem was renewed when new effective field theoretic techniques were developed for the study of the distribution of eigenfunction intensities $\mathcal{P}(|\psi|^2)$ of *random* Hamiltonians. As the disorder increases, these results predict that, the eigenfunctions become increasingly nonuniform, leading to an enhanced probability of finding anomalously large eigenfunction intensities in comparison with the random matrix theory (RMT) prediction. Thus, the notion of *prelocalized* states has been introduced [1–4] to explain the appearance of long tails in the distributions of the conductance and other physical observables [1].

Up to now all theoretical predictions [1–7] and numerical calculations [9,10] apply to disordered systems and are based on an ensemble averaging over disorder realizations. Their validity, however, for a quantum *dynamical* system (with a well defined classical limit) that behaves diffusively is not evident. Furthermore, based on an argument put forward in Ref. [6] (see also Ref. [15]), the far tail of $\mathcal{P}(|\psi|^2)$ is due to rare realizations of the disorder potential, and therefore requires an exponentially large number of eigenfunctions, which can only be accounted by disorder averaging. Here instead we study the statistical properties of eigenfunctions in a *dynamical* model without introducing any ensemble averaging. Our main conclusion is that in a generic dynamical system with classical diffusion, $\mathcal{P}(|\psi|^2)$ is described quite well by the nonlinear σ model (NLSM). We point out here that between the various theoretical works there is a considerable disagreement about the parameters that control the shape of $\mathcal{P}(|\psi|^2)$ and their dependence on time-reversal symmetry (TRS). More specifically, the NLSM suggests that the tail of $\mathcal{P}(|\psi|^2)$ in two dimensions (2D) is sensitive to TRS [4–6], while a direct optimal fluctuation (DOF) method predicts a symmetry independent result [7]. Recent numerical calculations [9] on the Anderson model seem to support the latter theory. This controversy was an additional motivation for the present work.

In the present Rapid Communication, we numerically study the distribution of intensities of the Floquet states of the kicked rotor (KR) on a torus [16] and its 2D generalization [17]. Our system is defined by the time-dependent Hamiltonian

$$H=H_0+kV\sum_m \delta(t-mT), \quad H_0(\{\mathcal{L}_i\})=\sum_{i=1}^{d}\frac{\tau_i}{2}(\mathcal{L}_i+\gamma_i)^2,$$

$$V(\{\theta_i\})=\cos(\theta_1)\cos(\theta_2)\cos(\alpha)+\frac{1}{2}\sin(2\theta_1)\cos(2\theta_2)\sin(\alpha), \quad (1)$$

where $\mathcal{L}_i$ denotes the angular momentum and θ_i the conjugate angle of one rotor. The kick period is T, k is the kicking strength, while τ_i is a constant inversely proportional to the moment of inertia of the rotor. The standard KR corresponds to $d=1$ (with $\theta_2=0$) whereas for $d=2$ we have a two-dimensional generalization. The parameter α breaks TRS [16], the parameters γ_i are irrational numbers whose meaning will be explained below. The Hamiltonian (1) describes a system that is kicked periodically in time and is integrable in the absence of the kicking potential. The motion generated by Eq. (1) is classically chaotic and for a sufficiently strong kicking strength k there is diffusion in momentum space with diffusion coefficient $D\equiv\lim_{t\to\infty}\langle\mathcal{L}^2(t)\rangle/t\simeq k^2/2T$ (within the random phase approximation) [16,17].

If the $\mathcal{L}_i$ are taken mod$(2\pi m_i/T\tau_i)$ where m_i are integers, Eq. (1) defines a dynamical system on a torus. The quantum mechanics of this system is described by a finite-dimensional time evolution operator for one period

$$U=\exp[-iH_0(\{\mathcal{L}_i\})T]\exp[-iV(\{\theta_i\})], \quad (2)$$

where we put $\hbar=1$. Upon quantization, additional symmetries associated with the discreteness of the momentum show up, which can be destroyed by introducing irrational values for the parameters γ_i's. The most striking consequence of quantization is the suppression of classical diffusion in momentum space due to quantum dynamical localization [16,17]. We introduce the eigenstate components $\mathbf{\Psi}_k(n)$ of the Floquet operator in the momentum representation by

$$\sum_m U_{mn}\Psi_k(n) = e^{i\omega_k T}\Psi_k(n). \tag{3}$$

The quantities ω_k are known as quasienergies, and their density is $\rho = T/2\pi$. The corresponding mean quasienergy spacing is $\Delta = 1/(\rho L^d)$, where L is the linear size of the system. The Heisenberg time is $t_H = 2\pi/\Delta$ while $t_D = L^2/D$ is the diffusion time (Thouless time). Now one can formally define a dimensionless conductance as $g = t_H/t_D = D_k L^{d-2}$ where $D_k = TD$ is the diffusion coefficient measured in the number of kicks. Four length scales are important here: the wavelength λ, the mean free path l_M, the linear extent of the system L, and the localization length ξ. According to Refs. [2–7] the field theoretical predictions are derived under the conditions

$$\lambda \ll l_M \ll L \ll \xi. \tag{4}$$

The first condition ensures that transport between scattering events may be treated semiclassically. This limit can be achieved for our system (1) when $k\to\infty$, $T\to 0$ while the classical parameter $K = kT$ remains constant. When $l_M \ll L$ as long as the motion is not localized (i.e., $L \ll \xi$) it is diffusive, since a particle scatters many times before it can traverse the system. The resulting mean free path for our system (1) is $l_M \simeq \sqrt{D_k}$ while the localization length for $d=1$ is $\xi \simeq D_k/2$ [16] and for $d=2$ is $\xi \simeq l_M e^{D_k/2}$ [17,18].

Here we calculate the distribution function $\mathcal{P}(t = L^d|\Psi_k(n)|^2)$ [19] by using a direct diagonalization of the Floquet operator (2). The TRS is broken entirely for $\alpha = 5.749$. In order to test the issue of dynamical correlations, we randomize the phases of the kinetic term of the evolution operator (2) and calculate the resulting $\mathcal{P}(t)$. This model will be referred to as random phase KR (RPKR). Since all our eigenfunctions have the same statistical properties (in contrast to the Anderson cases where one should pick up only eigenfunctions having eigenenergies within a small energy interval [9,10]) we make use of all of them in our statistical analysis. The classical parameter K is large enough in all cases to exclude the existence of any stability islands in phase space. The classical diffusion coefficient D_k is calculated numerically by iterating the classical map obtained from Eq. (1). Below we present our numerical results and compare them to the predictions of Refs. [2–7].

1D *kicked rotor*. It was shown in Ref. [15] that the effective field theory describing the semiclassical physics of the system is precisely the NLSM for quasi-one-dimensional metallic wires. Such a mapping, however, requires an averaging over an ensemble of rotors having the same classical limit. We point out again that in the calculations below we do not adopt such an averaging procedure.

The NLSM for quasi-1D systems can be solved exactly for the distribution function $\mathcal{P}_\beta(t)$, using a transfer matrix approach [3,5,6]. In the ballistic regime (where $g\to\infty$) RMT is applicable and one finds $\mathcal{P}^{RMT}_{(\beta=1)}(t) = \exp(-t/2)/\sqrt{2\pi t}$ and $\mathcal{P}^{RMT}_{(\beta=2)}(t) = \exp(-t)$ [20]. Here β denotes the corresponding Dyson ensemble [$\beta = 1(2)$ for preserved (broken) TRS]. As localization increases, the deviations from the RMT results of the body and the tails of the distribution $\mathcal{P}_\beta(t)$ become noticeable and can be parametrized by a single parameter that is the dimensionless conductance $g = D_k/L$.

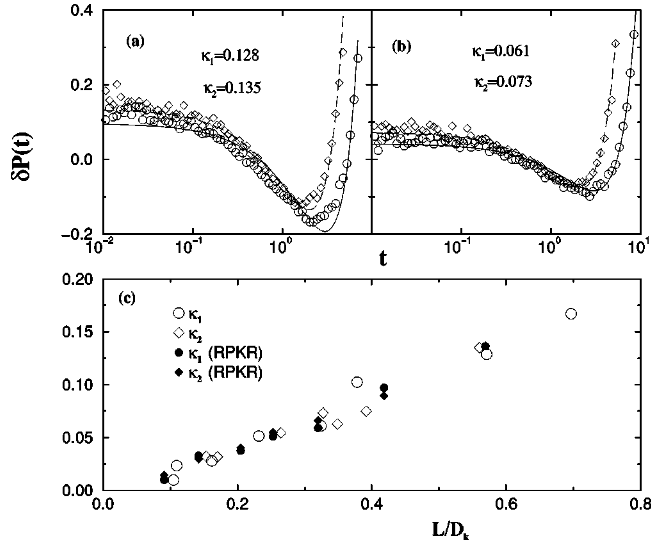

FIG. 1. Corrections to the distribution intensities $\delta\mathcal{P}_\beta(t)$ for the kicked rotator model, i.e., Eq. (1) for $d=1$. The system size is $L = 1024$, (○) $\beta=1$, (◇) $\beta=2$. The solid (dashed) lines are the best fit of Eq. (5) for $\beta=1(2)$ to the numerical data. (a) $D_k \approx 1800$ and (b) $D_k \approx 3150$; (c) shows the extracted diffusion propagator κ_β vs L/D_k.

For $t < \sqrt{D_k/L}$, according to all studies [3–5] $\mathcal{P}(t)$ is just the RMT result with polynomial corrections in powers of L/D_k, i.e., $\mathcal{P}_\beta(t) = \mathcal{P}^{RMT}_\beta(t)[1 + \delta\mathcal{P}_\beta(t)]$. The leading term of this expansion is given by

$$\delta\mathcal{P}_\beta(t) \simeq \kappa \begin{Bmatrix} 3/4 - 3t/2 + t^2/4 & \text{for } \beta=1 \\ 1 - 2t + t^2/2 & \text{for } \beta=2 \end{Bmatrix}, \tag{5}$$

where $\kappa \sim 1/g$ is the 1D diffusion propagator, which is identical for $\beta=1$ and $\beta=2$ since it is a classical quantity.

In Figs. 1(a) and 1(b) we report our numerical results for $\delta\mathcal{P}_\beta(t)$ for two representative values of D_k. One can clearly see that the agreement with the theoretical prediction (5) becomes better as D_k increases. This is due to the fact that by increasing D_k we are approaching the semiclassical region and therefore Eqs. (4) are better satisfied. At the same time higher order corrections in $\delta\mathcal{P}_\beta(t)$ become negligible with respect to the leading term given by Eq. (5). The resulting κ_1 and κ_2 obtained by the best fit of our data to Eq. (5) are found to be equal and in agreement with theory [see Fig. 1(c)]. We therefore conclude that in a generic dynamical system, the only parameter that controls the shape of the deviations $\delta\mathcal{P}_\beta(t)$ is the classical diffusion propagator. Moreover, our results are in excellent agreement with the recent NLSM predictions derived in the framework of diffusive disordered systems. Finally in Fig. 1(c) we also report the outcome of the RPKR model. The results remain essentially the same indicating that $\mathcal{P}_\beta(t)$ for quasi-1D systems are insensitive to dynamical correlations.

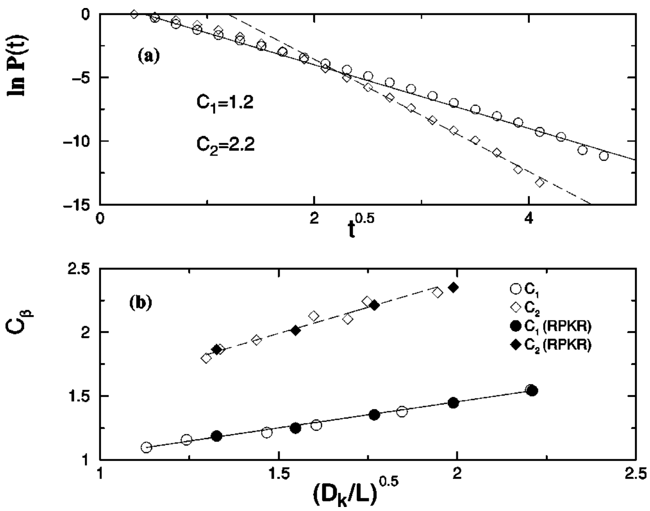

FIG. 2. (a) Tails of the distribution $\mathcal{P}_\beta(t>D_k/L)$ for the model (1) for $d=1$ with $L=1024, D_k\simeq 2625$ and for $\beta=1$ (○) and $\beta=2$ (◇). The solid (dashed) lines are the best fit of Eq. (6) for $\beta=1(2)$ to our data; (b) coefficients C_β vs $\sqrt{D_k/L}$. The solid (dashed) lines are the best fits to $C_\beta=A_\beta\sqrt{D_k/L}+B_\beta$ for $\beta=1(2)$.

The tail of the distribution $(t>D_k/L)$ deviates strongly from the RMT prediction and has a stretched exponential form [3–5]

$$\mathcal{P}_\beta(t)\simeq a_\beta \exp(-2C_\beta\sqrt{t}), \quad C_\beta=\beta\sqrt{D_k/L}, \qquad (6)$$

where a_β is a symmetry dependent constant. Our numerical results agree nicely with Eq. (6). In Fig. 2(a) we present an example of $\mathcal{P}_\beta(t)$. By fitting our data to Eq. (6) the coefficients C_1, C_2 can be extracted. In Fig. 2(b) we report the extracted stretched exponential coefficients C_β from the best fit of Eq. (6) as a function of the square root of the dimensionless conductance $g=D_k/L$. A nice linear behavior is observed. The best linear fit $C_\beta=A_\beta\sqrt{D_k/L}+B_\beta$ yields $A_{\beta=1}=0.41\pm0.05$ and $A_{\beta=2}=0.82\pm0.05$. The resulting ratio $R=A_2/A_1=2$ is in excellent agreement with the theoretical prediction (6). We have also calculated the stretched exponential coefficients C_β for the RPKR model. The results for various D_k values are summarized in Fig. 2(b) and show a nice agreement with the results obtained from the real Hamiltonian.

2D *kicked rotor*. According to Ref. [3], corrections to the body of $\mathcal{P}_\beta^{RMT}$ are still given by Eq. (5), but now κ is the 2D diffusion propagator.

Figures 3(a) and 3(b) show corrections to $\mathcal{P}_\beta^{RMT}$ for $g=D_k\gg 1$ for two representative values of D_k. We find again that the form of the deviations are very well described by Eq. (5) and the agreement becomes better for larger values of the diffusion constant. In Fig. 3(c) we summarize our results for various D_k values. The extracted κ_β values are obtained by the best fit of the data to Eq. (5). Again we find that κ_β depends linearly on $1/D_k$. However, contrary to the 1D-KR, here κ_1 and κ_2, are different. Moreover, the best fit with $\kappa_\beta=A_\beta D_k^{-1}+B_\beta$ yields $A_{\beta=1}=5.44\pm0.03$ and $A_{\beta=2}=10.84\pm0.04$ indicating that the ratio $R=A_2/A_1$ is close to

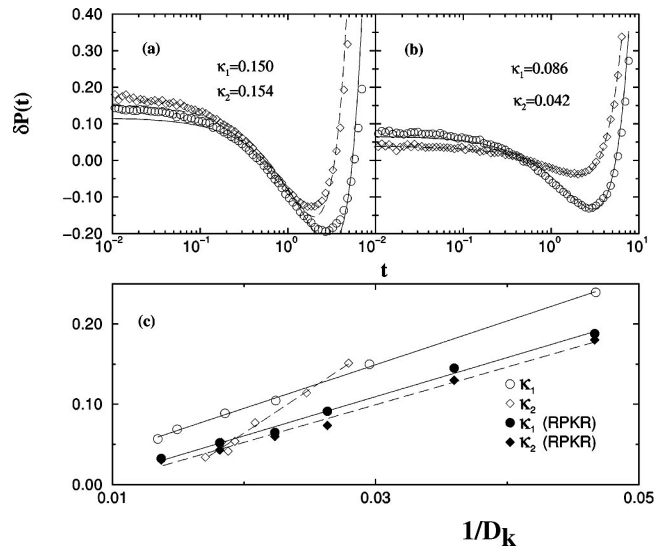

FIG. 3. Corrections to the distribution intensities $\delta\mathcal{P}_\beta(t)$ for the kicked rotator model (1) for $d=2$. The system size is $L=90$, (○) $\beta=1$, (◇) $\beta=2$. The solid (dashed) lines are the best fit of Eq. (5) for $\beta=1(2)$ to the numerical data. (a) $D_k\approx 34$ and (b) $D_k\approx 53$; (c) fit parameters κ_β vs D_k^{-1}. The solid (dashed) lines are the best fits to $\kappa_\beta=A_\beta D_k^{-1}+B_\beta$ for $\beta=1(2)$.

2, a value that could be explained on the basis of ballistic effects [6,9]. Taking the latter into account leads to an additional term in the classical propagator $\kappa_\beta=\kappa_{diff}+(\beta/2)\kappa_{ball}$. The first term is the one discussed previously and is associated with long trajectories that are of diffusive nature while the latter one is associated with short ballistic trajectories that are self-tracing [6]. Thus, when $\kappa_{diff}\ll\kappa_{ball}$ we get $R=2$. The calculation with the RPKR model shows, however, that the corresponding ratio is $R\simeq 1$, in

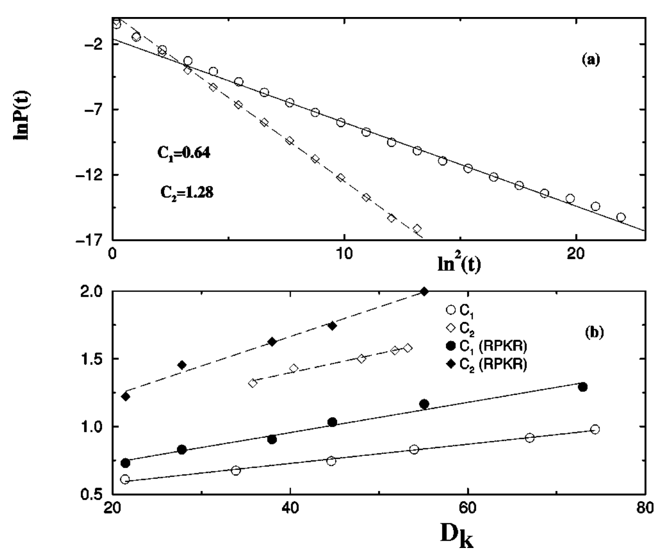

FIG. 4. (a) Tails of the distribution $\mathcal{P}_{\beta=1}(t>D_k)$ for the model (1) for $d=2$ and $D_k\simeq 35$. The system size is $L=80$, (○) $\beta=1$, (◇) $\beta=2$. The solid (dashed) lines are the best fit of Eq. (7) for $\beta=1(2)$ to the numerical data; (b) fitted log-normal coefficients C_β versus the classical diffusion coefficient D_k. The solid (dashed) lines are the best fits to $C_\beta=A_\beta D_k+B_\beta$ for $\beta=1(2)$.

agreement with the theoretical prediction for disordered systems with a pure diffusion. This indicates that dynamical correlations can be important in the 2D case.

For the tails of the distributions, the result of the NLSM within a saddle-point approximation [4,6] is

$$\mathcal{P}_\beta(t) \simeq \exp[-C_\beta^\sigma(\ln t)^2], \quad C_\beta^\sigma = \frac{\beta\pi^2\rho}{2}\frac{D}{\ln(L/l)}. \tag{7}$$

Note that the decay in the tails of Eq. (7) depends on β, as in the 1D-KR case [see Eq. (6)]. Recently, a DOF method was used to calculate the tails of $\mathcal{P}_\beta(t)$ [7]. It was found that the tails are still given by Eq. (7) but with a log-normal coefficient C that is independent of the parameter β,

$$C^{\mathrm{DOF}} = \pi^2\rho\frac{D}{\ln(L/\lambda)}. \tag{8}$$

Figure 4(a) shows a representative case of $\mathcal{P}_{\beta=1}(t>D_k)$. The tails show a log-normal behavior predicted by Eq. (7). In Fig. 4(b) we report the log-normal coefficients C_β extracted from the best fit to our numerical data, versus the classical diffusion coefficient. A pronounced linear behavior is observed in agreement with both theories. However, one clearly sees that C_1 differs from C_2 in contrast to the DOF prediction (8) and to recent numerical calculations done for the 2D Anderson model [9]. We point out here that in Ref. [9] the authors were not able to go to large enough values of conductance g (in comparison to our study) where the theory can really be tested. In contrast, the NLSM predicts a value of 2 for the ratio $R = C_2^\sigma/C_1^\sigma$. We note that C_β^σ is only the leading term in D_k. In order to calculate this ratio, we performed a fit to our data with $C_\beta = A_\beta D_k + B_\beta$. The resulting ratio was found to be $R = A_2/A_1 = 1.97 \pm 0.03$, in perfect agreement with the NLSM predictions. Finally, in Fig. 4(b) we also present our results for the RPKR model [using the same data as the one in Fig. 3(d)]. Again we found that the ratio $R = 1.96 \pm 0.03 \approx 2$. Thus $\mathcal{P}(t>D_k)$ depends on TRS and is described by the NLSM.

In summary, we have performed a detailed numerical analysis of the statistical properties of the wave function intensities $\mathcal{P}(t)$ of the standard KR on a torus and its 2D generalization. Based on these results, we concluded that the distribution $\mathcal{P}(t)$ of generic quantum *dynamical* systems with diffusive classical limit is affected by the existence of *prelocalized* states. The deviations from RMT are well described by field theoretical methods developed for disordered systems. In particular, in a clarifying way we have resolved the controversy between DOF and NLSM by demonstrating that the dependence of the tails of $\mathcal{P}_\beta(t)$ on TRS is described correctly by the latter theoretical approach.

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